Submodular Reranking with Multiple Feature Modalities for Image Retrieval [Supplementary Material]

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1 Proofs of PROPOSITIONS

1.1 Proof of PROPOSITION 1

Monotonicity

Proof. We have

$$F_m(\mathcal{S}) = H(\mathcal{V}_m \backslash \mathcal{S}) - H(\mathcal{V}_m \backslash \mathcal{S} | \mathcal{S}) = I(\mathcal{V}_m \backslash \mathcal{S}; \mathcal{S})$$

for graph $\mathcal{G}_m$, where $I(\mathcal{V}_m \backslash \mathcal{S}; \mathcal{S})$ is the mutual information between $\mathcal{V}_m \backslash \mathcal{S}$ and $\mathcal{S}$. As proved in [1], $I(\mathcal{V}_m \backslash \mathcal{S}; \mathcal{S})$ is monotonic when $|\mathcal{V}_m|$ is larger than $2|\mathcal{S}|$, which is the case in our framework. This completes the proof of the monotonicity property of $F_m(\mathcal{S})$.

Submodularity

Proof. We prove the submodularity by showing: for any $\mathcal{S}_1 \subset \mathcal{S}_2$ and a given example $a \in \mathcal{V}_m \backslash \mathcal{S}_2$, we have

$$F_m(\mathcal{S}_1 \cup \{a\}) - F_m(\mathcal{S}_1) \geq F_m(\mathcal{S}_2 \cup \{a\}) - F_m(\mathcal{S}_2)$$

We have

$$\begin{aligned}
&(F_m(\mathcal{S}_1 \cup \{a\}) - F_m(\mathcal{S}_1)) - (F_m(\mathcal{S}_2 \cup \{a\}) - F_m(\mathcal{S}_2)) \\
&= (H(a|\mathcal{S}_1) - H(a|\mathcal{V}_m \backslash \{\mathcal{S}_1 \cup a\})) \\
&\quad - (H(a|\mathcal{S}_2) - H(a|\mathcal{V}_m \backslash \{\mathcal{S}_2 \cup a\})) \\
&= (H(a|\mathcal{S}_1) - H(a|\mathcal{S}_2)) \\
&\quad + (H(a|\mathcal{V}_m \backslash \{\mathcal{S}_2 \cup a\}) - H(a|\mathcal{V}_m \backslash \{\mathcal{S}_1 \cup a\})) \\
&= H_1 + H_2
\end{aligned}$$

Since conditioning always reduces entropy, $H(a|\mathcal{S}_1) \geq H(a|\mathcal{S}_2)$, so that $H_1 \geq 0$. $\mathcal{V}_m \backslash \{\mathcal{S}_2 \cup a\} \subset \mathcal{V}_m \backslash \{\mathcal{S}_1 \cup a\}$, so that we have $H(a|\mathcal{V}_m \backslash \{\mathcal{S}_2 \cup a\}) \geq H(a|\mathcal{V}_m \backslash \{\mathcal{S}_1 \cup a\})$, leading to $H_2 \geq 0$. Therefore, $H_1 + H_2 \geq 0$, which completes the proof of the submodularity property of $F_m(\mathcal{S})$.

1.2 Proof of PROPOSITION 2

Monotonicity

Proof. We prove that $T(\mathcal{S})$ is monotonically increasing by showing $T(\mathcal{S}\cup\{a\}) \geq T(\mathcal{S})$, for all $a \in \mathcal{V}\backslash\mathcal{S}$ and $\mathcal{S} \subseteq \mathcal{V}$. Let $|\mathcal{S}|$ denote the cardinality of $\mathcal{S}$. Since items in $\mathcal{S}$ are ordered, we assume the rank of a in $\mathcal{S} \cup \{a\}$ as $r_a = |\mathcal{S}| + 1$ without loss of generality. We have

$$\begin{aligned}
&T(\mathcal{S} \cup \{a\}) - T(\mathcal{S}) \\
=&(1-q)\sum\nolimits_{s=1}^{|\mathcal{S}|+1} q^s \cdot \frac{1}{s} \sum\nolimits_{v_i,v_j \in \mathcal{S}\cup\{a\}, r_{v_i} < r_{v_j} = s} \mathcal{C}(v_i, v_j) \\
&-(1-q)\sum\nolimits_{s=1}^{|\mathcal{S}|} q^s \cdot \frac{1}{s} \sum\nolimits_{v_i,v_j \in \mathcal{S}, r_{v_i} < r_{v_j} = s} \mathcal{C}(v_i, v_j) \\
=&(1-q) \cdot q^{|\mathcal{S}|+1} \cdot \frac{1}{|\mathcal{S}|+1} \sum\nolimits_{v_i \in \mathcal{S}, r_{v_i} < r_a = |\mathcal{S}|+1} \mathcal{C}(v_i, a)
\end{aligned}$$

Since $\mathcal{C}(v_i, a) \geq 0$, $1 - q > 0$ and $q^{|\mathcal{S}|+1} > 0$, we can easily have $T(\mathcal{S} \cup \{a\}) - T(\mathcal{S}) \geq 0$ and $T(\emptyset) = 0$. This completes the proof of monotonically increasing property of $T(\mathcal{S})$.

Submodularity

Proof. We prove the submodularity by showing: for any $\mathcal{S}_1 \subset \mathcal{S}_2$ and a given example $a \in \mathcal{V}\backslash\mathcal{S}_2$, we have

$$T(\mathcal{S}_1 \cup \{a\}) - T(\mathcal{S}_1) \geq T(\mathcal{S}_2 \cup \{a\}) - T(\mathcal{S}_2)$$

From the derivation for monotonicity, we have

$$\begin{aligned}
&T(\mathcal{S}_1 \cup \{a\}) - T(\mathcal{S}_1) \\
=&(1-q) \cdot q^{|\mathcal{S}_1|+1} \cdot \frac{1}{|\mathcal{S}_1|+1} \sum\nolimits_{v_i \in \mathcal{S}_1, r_{v_i} < r_a = |\mathcal{S}_1|+1} \mathcal{C}(v_i, a)
\end{aligned}$$

and

$$\begin{aligned}
&T(\mathcal{S}_2 \cup \{a\}) - T(\mathcal{S}_2) \\
=&(1-q) \cdot q^{|\mathcal{S}_2|+1} \cdot \frac{1}{|\mathcal{S}_2|+1} \sum\nolimits_{v_i \in \mathcal{S}_2, r_{v_i} < r_a = |\mathcal{S}_2|+1} \mathcal{C}(v_i, a)
\end{aligned}$$

For notational simplicity, we let $n_1 = |\mathcal{S}_1| + 1$ and $n_2 = |\mathcal{S}_2| + 1$. Define

$$\begin{aligned}
k_1 &= \frac{1}{n_1} \sum\nolimits_{v_i \in \mathcal{S}_1, r_{v_i} < r_a = n_1} \mathcal{C}(v_i, a) \\
k_2 &= \frac{1}{n_2} \sum\nolimits_{v_i \in \mathcal{S}_2, r_{v_i} < r_a = n_2} \mathcal{C}(v_i, a)
\end{aligned}$$

as the average relative ranking measure between a and all items in $\mathcal{S}_1$ and $\mathcal{S}_2$, respectively. Then k_1 and k_2 can be represented as

$$k_2 = \frac{1}{n_2}(n_1 k_1 + \sum\nolimits_{v_i \in \mathcal{S}_2 \setminus \mathcal{S}_1, r_{v_i} < r_a = n_2} \mathcal{C}(v_i, a))$$

Suppose $|\mathcal{S}_2| = |\mathcal{S}_1| + n$, according to Eq. 6 in the paper, $\mathcal{C}(v_i, a)$ can be considered as a random variable $\phi \in [0, 1]$, so that we have $k_2 = \frac{1}{n_2}(n_1 k_1 + \sum_n \phi)$, where the upper bound of $\sum_n \phi$ is nk_1. Hence

$$\begin{aligned} &(T(\mathcal{S}_1 \cup \{a\}) - T(\mathcal{S}_1)) - (T(\mathcal{S}_2 \cup \{a\}) - T(\mathcal{S}_2)) \\ =&(1-q) \cdot q^{|\mathcal{S}_1|}(k_1 - q^n k_2) \end{aligned}$$

Since $(1 - q) > 0$ and $q^{|\mathcal{S}_1|} > 0$, we only need to prove $k_1 - q^n k_2 \geq 0$. Let $k_1 - q^n k_2 = k_1 - q^n \frac{n_1 k_1 + \sum_n \phi}{n_2}$, which reaches its minimum when $\sum_n \phi$ reaches its upper bound. In this case, we have

$$k_1 - q^n k_2 = k_1 - q^n \frac{n_1 k_1 + n k_1}{n_2} = k_1(1 - q^n) \geq 0$$

This completes the proof of submodularity property of $T(\mathcal{S})$.

References

1. Nemhauser, G.L., Wolsey, L.A., Fisher, M.L.: An analysis of approximations for maximizing submodular set functions. Mathematical Programming **14** (1978) 265–294